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The action of the group of bundle-automorphisms on the space of connections and the geometry of gauge theories

P. COTTA RAMUSINO (*) and C. REINA (**)

Dipartimento di Fisica dell'Università
Via Celoria 16 - Milano (Italy)

Abstract. *Some aspects of the geometry of gauge theories are sketched in this review. We deal essentially with Yang-Mills theory, discussing the structure of the space of gauge orbits and the geometrical interpretation of ghosts and anomalies. Occasionally we deal also with classical «gauge theories» of gravitation and in particular we study the action of the group of diffeomorphisms on the space of linear connections. Finally we comment on the mathematical interpretation of anomalies in field theories.*

INTRODUCTION

From the mathematical point of view the most natural set up for a classical theory of gauge fields requires the following data:

1. A geometric environment, that is a principal fibre bundle $P \xrightarrow{G} M$, with structure group G, on a differentiable manifold M, and some vector bundles E_i associated to P. Usually some further geometrical structures on P, E_i or M have

(*) Istituto Nazionale di Fisica Nucleare - Sezione di Milano.

(**) Gruppo Nazionale di Fisica Matematica del C.N.R.

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to be assigned to construct the theory (e.g. a fixed Riemannian structure on M, or a fibre metric on E, etc.).

2. A space of fields, that is the space $\mathscr{C}$ of connections on P (gauge fields) and the spaces F_i of sections of E_i (Higgs fields). These spaces will be given later the structure of Hilbert manifolds.

3. A differentiable function $S : \mathscr{C} \times \oplus_i F_i \to \mathbb{R}$ playing the role of the action of the theory.

Given these data, the invariance group of the theory is the subgroup of the group of the automorphisms of the geometric structure which, acting on the space of fields, preserves the action functional S.

The main purpose of these lectures is to study the action of the invariance group on the space of fields. To be definite, we shall be concerned with pure gauge theories, i.e. we shall limit ourselves to study the action of the invariance group on the space of connections $\mathscr{C}$.

We will also occasionally consider classical gauge theories of gravitation, where we still have a suitable principal bundle P and a space of connections $\mathscr{C}$. The action functional for these theories involves, in general, other fields, as a «vierbein» or a «metric», and we have to study the action of the appropriate invariance group on the product of the space of connections with the space of the other geometrical fields which are present in the theory.

Some parts of the paper contain basic definitions of differential geometry and gauge theories. We hope that, in this way, the paper will be more readable to both mathematicians and physicists.

1. GEOMETRIC SET UP FOR GAUGE THEORIES

1.1. Invariance Groups in Gauge Theories

The invariance group of a gauge theory is a subgroup of the group of automorphisms of the principal fibre bundle $P \xrightarrow{\pi} M$ on which the theory is based. The automorphisms group is, by definition, the subgroup of the group Diff (P) of diffeomorphisms of P, which preserve the fibration, i.e.

$$\mathrm{Aut}\,(P) = : \{f \,|\, f \in \mathrm{Diff}\,(P) \text{ s.t. } \forall a \in G, u \in P \text{ then } f(u \cdot a) = f(u) \cdot a\}$$

where $(u, a) \rightsquigarrow u \cdot a$ denotes the right action of G on P.

For any $f \in \mathrm{Aut}\,(P)$, there exists a unique diffeomorphism (jf) of M which makes the diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & P \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow[jf]{} & M \end{array}$$

commute; i.e. $(jf)(\pi u) = \pi f(u)$. The map $j: \mathrm{Aut}\,(P) \to \mathrm{Diff}\,(M)$ given by $f \leadsto jf$ is easily seen to be a group homomorphism (1). Its kernel is a subgroup of Aut (P), which is denoted by the symbol $\mathrm{Aut}_V(P)$ and is called the group of vertical automorphisms of P. That is, we set:

$$\mathrm{Aut}_V(P) =: \{f \mid f \in \mathrm{Aut}\,(P) \text{ s.t. } jf = \text{identity} \in \mathrm{Diff}\,(M)\}.$$

From the definitions above, it is clear that we have an exact sequence of groups, which, restricted to the identity connected components, reads

$$1 \longrightarrow \mathrm{Aut}_V(P) \longrightarrow \mathrm{Aut}\,(P) \longrightarrow \mathrm{Diff}\,(M) \longrightarrow 1.$$

For more details on these groups see e.g. Trautman [1].

As we mentioned before, in order to construct the theory, i.e. to define an action functional S on the space of gauge fields, one usually needs further geometric structures on the principal fiber bundle. The restricted automorphism group $\overline{\mathrm{Aut}\,(P)}$ of P (with the additional structures) is the subgroup of Aut P which preserves (those structures and hence) the action.

We consider now the group $\overline{\mathrm{Aut}\,P}$ in two examples: Yang-Mills theories and gravitational theories.

Example 1.1.1

In Yang-Mills theory over the Minkowski space-time M, the Minkowski metric η on M has to be preserved. This reduces Diff (M) to the Poincarè group in the sequence above, yielding

$$1 \longrightarrow \mathrm{Aut}_V(P) \longrightarrow \overline{\mathrm{Aut}\,(P)} \longrightarrow \mathrm{ISO}\,(1,3) \longrightarrow 1.$$

So $\overline{\mathrm{Aut}\,(P)}$ is an extension of ISO (1,3) by $\mathrm{Aut}_V(P)$. If one studies the euclidean version of Yang-Mills theory, ISO (1,3) in the sequence above, has to be replaced by ISO (4).

More generally one knows that only a conformal class of metric has to be given on a four-dimensional manifold in order define the action functional for pure Yang-Mills theories. So one can study the theory on the conformally compactified Minkowsky space or on S^4 in the euclidean case. Accordingly the invariance group is an extension of the appropriate conformal group (i.e. $SO(2,4)/\mathbb{Z}_2$ or $SO(1,5)/\mathbb{Z}_2$) by the group $\mathrm{Aut}_V(P)$.

Example 1.1.2

One of the most natural consequences of the equivalence principle is that the

(1) For the moment we give all the groups the C^∞ topology, and restrict our attention to the components connected with the identity.

gravitational forces have to be considered as a connection on the space-time. In our picture, this requires that the principal fibre bundle of gravitational theories has to be the linear frame bundle $L(M)$ of M. We stress that $L(M)$ has to be considered as a frame bundle and not merely as a principal fibre bundle, that is the *soldering form* ϑ of $L(M)$ is an «absolute element» of the theory and has to be preserved (for more details see Trautman [1]). Notice that this is true independently of the action which is assumed in the theory. A diffeomorphism ψ of M naturally induces an automorphism of $L(M)$ via its tangent map $\psi_* : T_xM \to T_{\psi(x)}M$. Any $u \in L(M)$ is a frame in $T_{\pi(u)}M$, i.e. it is a linear isomorphism $u : \mathbb{R}^n \to T_{\pi(u)}M$, where $\pi : L(M) \to M$ is the canonical projection. Accordingly, the composition

$$\psi_* |_{\pi(u)} \circ u \ : \mathbb{R}^n \longrightarrow T_{\psi(\pi(u))}M$$

is a new frame u' at $\psi(\pi(u))$. We call then the natural lift $\ell(\psi)$ of a diffeomorphism ψ, the automorphism of LM given by: $\ell(\psi) =: \psi_* |_{\pi(u)} \circ u$. We know from Kobayashi and Nomizu ([2], vol. 1, pag. 229) that for any automorphism f of $L(M)$ the following conditions are equivalent:

i) f is the natural lift of a diffeomorphism ψ of M, i.e. $f = \ell(\psi)$

ii) f leaves the soldering form invariant, i.e. $f^* \vartheta = \vartheta$.

Since the lifting map $\ell : \mathrm{Diff}(M) \to \mathrm{Aut}(L(M))$ splits the exact sequence of automorphism groups of $L(M)$, we have:

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Aut}_V(L(M)) \arrow[r] & \mathrm{Aut}(L(M)) \arrow[r, "j"] & \mathrm{Diff}(M) \arrow[r] \arrow[l, bend left, "\ell"] & 1 \end{tikzcd}$$

and $j \circ \ell =$ identity. Accordingly appropriate sequence for theories on soldered bundles becomes

$$1 \longrightarrow 1 \longrightarrow \ell(\mathrm{Diff}(M)) \longrightarrow \mathrm{Diff}(M) \longrightarrow 1$$

because vertical automorphism other than the identity do not preserve the soldering form. Incidentally notice that $\ell(\mathrm{Diff}(M))$ and $\mathrm{Diff}(M)$ are isomorphic.

Obviously any action functional for a theory of gravitation has to be invariant under diffeomorphisms.

By comparing these examples, one sees that there exists a remarkable difference between theories on soldered principal fibre bundles and theories on generic principal fibre bundles, as no vertical automorphism of the bundle belongs to the invariance group in the first case.

1.2. The gauge group

In Yang-Mills gauge theories, the gauge group is by definition the group $\mathrm{Aut}_V(P)$ and this is the group we want to study in detail. The gauge group is a subgroup of the invariance group for Yang-Mills theories. First of all we will show that $\mathrm{Aut}_V(P)$ is isomorphic to a group $\mathcal{G}$ of sections of a bundle over M, in this way the elements of the gauge group will look more «similar» to the gauge transformations considered by physicists and it will be easier for us to construct a Hilbert-Lie structure on $\mathcal{G}$ later on in this section.

Consider the bundle $A\mathrm{d}P =: P \times_{\mathrm{Ad}} G$, associated to P via the adjoint action of G on itself (2). This is a bundle of groups, but it is not a principal fibre bundle, so it can have plenty of sections even when P is not trivial. The set $\Gamma(A\mathrm{d}P)$ of smooth sections of $A\mathrm{d}P$ is clearly a group under pointwise multiplication, which will be denoted by $\mathcal{G}$. We have now the following

THEOREM 1.2.1. *The Group $\mathcal{G}$ is isomorphic to* $\mathrm{Aut}_V(P)$.

Proof. If $f \in \mathrm{Aut}_V(P)$, then u and $f(u)$ belong to the same fibre of P. Hence there exists a map $\tilde{g} : P \to G$ such that $f(u) = u \cdot \tilde{g}(u)$. The equivariance of f, i.e. the fact that $f(u \cdot a) = f(u)a$, implies that $(u \cdot a) \cdot \tilde{g}(u \cdot a) = u \cdot (\tilde{g}(u)a)$. Accordingly one must have $\tilde{g}(ua) = a^{-1}\tilde{g}(u)a$. Obviously the set $\tilde{G}$ of such maps $\tilde{g} : P \to G$ is a group under pointwise multiplication. Notice also that, since

$$f_2(f_1(u)) = f_2(u \cdot \tilde{g}_1(u)) = f_2(u) \cdot \tilde{g}_1(u) = u \cdot (\tilde{g}_2(u)\, \tilde{g}_1(u))$$

the correspondence $f \leadsto \tilde{g}$ is a group isomorphism.

Next, to any $\tilde{g} \in \tilde{G}$ we associate a section g of $A\mathrm{d}P$ by simply taking the quotient of the graph $(u, g(u)) \subset P \times G$ of g with respect to the adjoint action of G. Conversely, to any $g \in \mathcal{G}$, we associate $\tilde{g} \in \tilde{G}$ by $\tilde{g}(u) =: u^{-1}[g(\pi(u)]$ where $u \in P$ is considered as a map $u[\ \ ] : G \to A\mathrm{d}P|_{\pi(u)}$, (see [2], proposition 5.4, vol. 1, pag. 55). Obviously enough $u[\ \ ]$ is a group isomorphism, so $\mathcal{G}$ and $\tilde{G}$ are ismorphic.

From now on we will refer to $\mathcal{G}$ as the *gauge group* itself. Note that, for the time being, $\mathcal{G}$ is an abstract group, so strictly speaking, it does not make any sense to speak about its Lie algebra. It is however convenient to introduce a Lie algebra of smooth sections of a suitable bundle which, at it will be explained later,

(2) I.e. we consider the quotient of $P \times G$ under the action of G given by $((u, a), b) \leadsto (u \cdot b, b^{-1}ab)$, where $u \in P$ and $a, b \in G$.

plays the role of the Lie algebra of $\mathscr{G}$.

Consider the space $\Omega^*(M, adP)$ of smooth section of the bundle $\Lambda^*M \otimes adP$. Here $adP = P \times_{ad} G'$, where G' is the Lie algebra of G and the action of G on its Lie algebra is the adjoint action (3). Elements of $\Omega^*(M, adP)$ are forms on M whose values at $x \in M$ belong to the fibre of adP at x, i.e. their local expressions are G'-valued forms. Consider now elements $\omega_1, \omega_2 \in \Omega^*(M, adP)$ s.t. $\omega_1 = \alpha_1 \otimes \tau_1$, $\omega_2 = \alpha_2 \otimes \tau_2$. Again any element $u \in P$ can be seen as a map $u : G' \to adP|_{\pi(u)}$ ([2] proposition 5.4, vol. I, p. 55). Then we can define

$$[\omega_1, \omega_2](x) = : \alpha_1 \wedge \alpha_2 \otimes (u[u^{-1}\tau_1, u^{-1}\tau_2]_{G'}) \tag{1.2.2}$$

where $x = \pi(u)$ and the bracket in the r.h.s. is the bracket of G'.

The bracket (1.2.2) gives $\Omega^*(M, adP)$ the structure of a graded Lie Algebra. For, if ω^p, ω^q are respectively p- and q-forms in $\Omega^*(M, adP)$, then we have:

$$[\omega^p, \omega^q] = (-1)^{pq+1}[\omega^q, \omega^p]$$

and the graded Jacobi identity

$$[\omega^p, [\omega^q, \omega^r]] = [[\omega^p, \omega^q], \omega^r] + (-1)^{pq}[\omega^q, [\omega^p, \omega^r]]$$

holds.

In particular $\Omega^0(M, adP) = \Gamma(adP)$ has a natural structure of a Lie algebra and it is intuitively clear that it should be something like the «Lie algebra» of the gauge group $\mathscr{G}$. In order to make this statement mathematically precise, we need to give a Lie structure to the group $\mathscr{G}$. Consider now any vector bundle F over M. For any positive integer k we can introduce the k-Sobolev norm on the vector-space of C^∞-sections of F and consider its completion $H^k(F)$ which is an Hilbert space (see [3], [4], [5]). We always assume $k > \dim M/2$ in order to guarantee the validity of the Sobolev embedding theorem.

Accordingly we denote by $\Omega^p_k(M, adP)$ the k-Sobolev completion of $\Omega^p(M, adP)$ for any p.

The gauge group itself $\mathscr{G}$ can be considered as a subset of the space of sections of a vector bundle which has a suitable space of complex matrices as standard fiber; so it can be shown that $\mathscr{G}$ itself can be «enlarged» to an Hilbert Lie group $\mathscr{G}_k$ (see [3]).

We have now the following:

PROPOSITION (1.2.3). *The space $\Omega^0_k(M, adP)$ is the Lie algebra $\mathscr{G}'_k$ of the Hilbert*

(3) The elements of $\Omega^p(M, adP)$ can be considered as tensorial forms on P of type ad ([2] Esample 5.2 vol. I, p. 76).

Lie group $\mathcal{G}_k$.

Proof. For the proof see Mitter and Viallet [3].

We will be also interested in the following two subgroups of $\mathcal{G}_k$. The first one is the group

$$\overline{\mathcal{G}_k} = \mathcal{G}_k / Z$$

where Z is the centre of $\mathcal{G}_k$, which is nothing else than the group of sections of AdP having values in the centre of G. If G has discrete centre, as we always assume in physical applications, $\mathcal{G}_k$ is a covering of $\overline{\mathcal{G}_k}$. So their Lie algebras are isomorphic.

The second subgroup of $\mathcal{G}_k$ we are interested in is the so called pointed gauge group $\mathcal{G}_k^0$. This is defined as follows

$$\mathcal{G}_k^0 = \{g \in \mathcal{G}_k \mid g(x_0) = e \in G\},$$

where x_0 is a fixed point of M (i.e. the point «at infinity»). The Lie algebra of $\mathcal{G}_k^0$ is given by

$$\mathcal{G}_k^{0\prime} = \{\xi \in \Omega_k^0(M, adP) \mid \xi(x_0) = 0\}.$$

1.3. The group of diffeomorphisms

As we have seen in section 1.1, the invariance group which is appropriate for gravitational theories is the group Diff (M). This can be conveniently described as follows.

Let $F = M \times M$ be the trivial fiber bundle on M, having M as standard fibre. Sections of F are maps $f: M \to M$. We restrict our attention to the set $C^1(F)$ of sections of class C^1, the topology of $C^1(F)$ will be the topology of «uniform convergence» up to the first derivative. The group of diffeomorphisms (of class C^1) is then

$$C^1\mathcal{D} = \{f \in C^1(F) \mid \exists f^{-1} \text{ and } f^{-1} \in C^1(F)\}.$$

It is easy to show that $C^1\mathcal{D}$ is open in $C^1(F)$.

Next we want to give a Hilbert manifold structure to this group. This can be done again by resorting to Sobolev spaces of maps. Let $H^k(F)$ be the Hilbert manifold of sections of F in Sobolev class k. Then for $k > 1 + \dim M/2$, let

$$\mathcal{D}_k = H^k(F) \cap C^1\mathcal{D}.$$

Due to the Sobolev embedding theorem, the inclusion $H^k(F) \subseteq C^1\mathcal{D}$ is continuous and hence $\mathcal{D}^k$ is open in $H^k(F)$. Accordingly $\mathcal{D}^k$ itself is a Hilbert manifold.

Collecting the results of Ebin, we have the following

PROPOSITION 1.3.1. *For* $k > 1 + \dim M/2$ *we have:*
i) $\mathcal{D}^k$ *is a group under composition of maps*
ii) *For any* $\psi \in \mathcal{D}^k$, *the right action* $R_\psi : \mathcal{D}^k \to \mathcal{D}^k$ *given by* $R_\psi \phi = \phi \circ \psi$ *is smooth.*
iii) *For any* $\phi \in \mathcal{D}^k$ *the left action* $L_\phi : \mathcal{D}^k \to \mathcal{D}^k$ *given by* $L_\phi \psi = \phi \circ \psi$ *is continuous, but not differentiable.*
iv) $\mathcal{D}^k$ *is a topological group.*

Proof. See Ebin [4].

The lack of differentiability of the left action of $\mathcal{D}^k$ on itself means that $\mathcal{D}^k$ is not a Lie group. Novertheless it is a Hilbert manifold, and one can consider the tangent space $T_e\mathcal{D}^k$ at the identity $e \in \mathcal{D}^k$. In order to have a better understanding about the structure of the space $T_e\mathcal{D}^k$, let us consider the one parameter families of diffeomorphisms $\psi_t \in \mathcal{D}^k$ such that $\psi_0 = e$. Germs of ψ_t at $t = 0$ can be identified with vector fields on M of Sobolev class k, that is we have $T_e\mathcal{D}^k = H^k(TM)$. Since in our hypotheses $k > 1 + \dim M/2$, elements of $H^k(TM)$ are at least C^1 vector fields on M. Naturally one would like to define the bracket for two elements in $T_e\mathcal{D}^k$ as the bracket for the corresponding vector fields in $H^k(TM)$. Unfortunately if $X, Y \in H^k(TM)$ then $[X, Y] \in H^{k-1}(TM)$, so for any k, $T_e\mathcal{D}^k$ is not closed under the bracket induced by $H^k(TM)$ and so it is not a Lie Algebra.

To get a Lie algebra one has to take the limit for $k \to \infty$, i.e. one has to consider smooth vector fields on M. The corresponding group of smooth diffeomorphisms however will not be a Hilbert manifold any more, since the model space will be a Frechet space; actually, $\mathrm{Diff}(M) = \mathcal{D}^\infty$ is a projective limit of Hilbert spaces in the sense of Omori [6] (4).

In the following we shall work with a subgroup of $\mathcal{D}^k$. We say that a diffeomor-

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(4) Recall the following definition: A topological space V is a strong inverse limit of Hilbert manifolds (i.e. a strong ILH - manifold) modelled over the Frechet space $\mathcal{F}$ if

i) V is the projective limit of smooth Hilbert manifolds V_i ($i \in \mathbb{Z}^+$) modelled over a sequence of Hilbert spaces E_i with projective limit $\mathcal{F}$, and $V_i \supset V_j$ for $j \geqslant i$.

ii) For any $x \in V$ there exists a coordinate neighborhood $\mathcal{U}_i(x)$ in V_i and homomorphisms $\psi_i : \mathcal{U}_i(x) \to \mathcal{A}_i$, where $\mathcal{A}_i$ is an open subset of E_i, such that $\mathcal{U}_i(x) \supset \mathcal{U}_j(x)$ for $j \geqslant i$, and $\psi_i|_{\mathcal{U}_i} = \psi_j$.

iii) The projective limit $\varprojlim \mathcal{U}_i(x)$ is an open neighborhood of x in V.

phism $\psi \in C^1\mathscr{D}$ strongly fixes a point $x_0 \in M$ if $\psi(x_0) = x_0$ and $\psi_*|_{x_0} =$ identity on $T_{x_0}M$. For $k > 1 + \dim M/2$ we define the groups of diffeomorphisms strongly fixing a point x_0 of Sobolev class k as:

$$\mathscr{D}_0^k = \{\psi \mid \psi \in \mathscr{D}^k,\ \psi(x_0) = x_0 \ \text{ and } \ \psi_*|_{x_0} = \text{id} \ \text{ on } T_{x_0}M\}.$$

As for $\mathscr{D}^k$, one can show that is a topological group, and also a Hilbert manifold, but these two structures do not «fit» together to yield a Lie group structure on $\mathscr{D}_0^k$, for any k. The tangent space $T_e\mathscr{D}_0^k$ at the identity $e \in \mathscr{D}_0^k$ is easily described in terms of vector fields on M. We have $T_e\mathscr{D}_0^k = H_0^k(TM)$ where:

$$H_0^k(TM) = \{X \mid X \in H^k(TM), X(x_0) = 0, T_{x_0}X = 0\},$$

where TX is the tangent map to X.

We now lift the action of these groups to let them act on the linear frame bundle via the natural lift $\mathfrak{L}$ of section 1.2. We have the following

PROPOSITITION (1.3.2). *Let* $k > 1 + \dim M/2$, *then*

i) $\forall \psi \in \mathscr{D}_0^k$, *the natural lift* $\mathfrak{L}(\psi) : L(M) \to L(M)$ *is such that*

$$\mathfrak{L}(\psi)(u) = u$$

for any $u \in \pi^{-1}(x_0)$

ii) $\forall X \in T_e\mathscr{D}_0^k$, *the natural lift* $\mathfrak{L}_*X$ *is such that*

$$\mathfrak{L}_*X|_u = 0$$

for any $u \in \pi^{-1}(x_0)$.

Proof. Let ψ_t be a 1-parameter subgroup of $\mathscr{D}_k^0$, with $\psi_0 = e$. Then

$$\mathfrak{L}(\psi_t)(u) = \psi_{t*}|_{\pi(u)} \circ u \in \pi^{-1}(\psi(\pi(u)));$$

now if $u \in \pi^{-1}(x_0)$, we have $\psi(\pi(u)) = \pi(u)$ and $\psi_*|_{\pi(u)} = \text{id}$, from which (i) follows. Next, by taking the germ of $\mathfrak{L}(\psi_t)$ at $u \in \pi^{-1}(x_0)$, it is easy to show that (ii) holds. Note that the proof requires that ψ is at least of class C^1, which is guaranteed as far as $k > 1 + \dim M/2$.

1.4. The action of the invariance groups on connections

Let $\mathscr{C}$ be the space of smooth connections on P. Each element $A \in \mathscr{C}$ can be represented as a pseudotensorial one-from of type ad on P ([2], vol. 1, pag. 77). Accordingly the difference $A' - A$ of two connections is a tensorial one form of type ad. It is well known (see [2], vol. 1, Example 5.2) that the space of such tensorial form is isomorphic to the space $\Omega^1(M, adP)$, and we shall not distinguish between these two spaces.

So $\mathscr{C}$ is an affine space modelled over $\Omega^1(M, adP)$. Any automorphism $f : P \to P$ induces a transformation on $\mathscr{C}$ which is simply given by the pull-back of connection forms (see [2], vol. I, pag. 81). Then we have an action:

$$(1.4.1) \qquad \begin{aligned} \mathrm{Aut}\,(P) \times \mathscr{C} &\longrightarrow \mathscr{C} \\ (f, A) &\rightsquigarrow f^*A. \end{aligned}$$

Let now f_t be a differentiable one parameter family of automorphisms of P, and let Z be the vector field on P generating f_t. For any $A \in \mathscr{C}$, we have a curve $A_t = f_t^*A$ and a corresponding curve $\tau_t = A_t - A$ in $\Omega^1(M, adP)$. Then we compute

$$\frac{\mathrm{d}\tau_t}{\mathrm{d}t}\Big|_0 = \mathscr{L}_Z A$$

where $\mathscr{L}_Z$ denotes the Lie derivative along Z. Accordingly we have a linear map

$$\alpha : \mathrm{aut}\,(P) \rightsquigarrow \Gamma(T\mathscr{C})$$

given by $\alpha(Z) = \mathscr{L}_Z A$ where $\mathrm{aut}\,P$ the subalgebra of vector fields on P which generate flows of automorphisms of P.

Next we need an explicit representation for the linear map α. We have the following:

PROPOSITION (1.4.2). *Let $Z \in \mathrm{aut}\,(P)$, $A \in \mathscr{C}$; then*

$$\mathscr{L}_Z A = \mathrm{d}_A(Y) + h(Z) \lrcorner F$$

where $Y = : A(Z) \in \Omega^0(M, adP)$ is related to the vertical component of Z, $h(Z)$ is the A-horizontal component of Z, F is the curvature of A and d_A is the covariant derivative, which is a linear first order operator from $\Omega^0(M, adP)$ to $\Omega^1(M, adP)$.

Proof. See e.g. Trautman [7].

Notice that $Z \in \mathrm{aut}\,P$ implies that $A(Z) \in \Omega^0(M, adP)$.

We now consider two particularly interesting cases:

i) Vertical automorphisms.

Let f_t be a one parameter family of vertical automorphisms of P. As we already know the vector field Z generating f_t is a section of the vertical tangent bundle to P. Hence $h(Z) = 0$, and we have

$$(1.4.3) \qquad \mathscr{L}_Z A = \mathrm{d}_A A(Z).$$

ii) Lift of diffeomorphisms for frame bundles.

When $P = L(M)$ we can consider lift of diffeomorphisms. For any vector field X on M, let $Z = \ell_* X$ be its lift to $L(M)$. Now the vertical and horizontal parts of Z defined with respect to a connection A entirely depend on X. In particular the correspondence

$$X \rightsquigarrow A(Z)$$

is a first order differential operator (see Trautman [5]) so that the map

$$X \rightsquigarrow \mathcal{L}_{\ell_* X} A \tag{1.4.4}$$

is a second order linear differential operator.

One can show by direct computation that both the operators d_A and $\mathcal{L}_{\ell_x}$ have injective symbols.

In the following we would like to give a Hilbert manifold structure also to the spaces of connections. This is done by considering the k-Sobolev norm ($k > \dim M/2$) for the space $\Omega^1(M, adP)$. By completion one gets Hilbert spaces $\Omega^1_k(M, adP)$ and the space $\mathscr{C}_k$ of k-Sobolev connections, will be defined as an affine space modelled on $\Omega^1_k(M, adP)$.

2. GAUGE ORBIT SPACES

2.1. Some physical motivations

At this point it is useful to interrupt briefly our mathematical description to sketch some of the features of gauge theories that motivate our study.

In the Feynman approach to (euclidean) quantum physics one should consider expressions like:

$$\frac{\displaystyle\int_{\mathscr{C}} e^{-S(A)}\, \mathscr{F} DA}{\displaystyle\int_{\mathscr{C}} e^{-S(A)} DA} \tag{2.1.1}$$

where the Feynman integrals are computed over the space $\mathscr{C}$ of connections with measure DA. Here $\mathscr{F}$ is a gauge invariant functional over $\mathscr{C}$ and S is, as usual, the euclidean action. So the integrands in (2.1.1) are actually defined in the space of gauge orbits, i.e. in the quotient space $\mathscr{C}/\mathscr{G}$.

Since the orbits in $\mathscr{C}$ are expected to have infinite measure one should really integrate the integrands in (2.1.1) over $\mathscr{C}/\mathscr{G}$ instead of $\mathscr{C}$, but this is difficult.

What one usually does is to choose a section, i.e. a map $\sigma : \mathscr{C}/\mathscr{G} \to \mathscr{C}$. This «change of variables» gives a «Jacobian determinant» (the so called Faddeev--Popov determinant) that will be denoted by «$\det M$». As we will see later in more details, M is an operator which maps the (infinite dimensional) Lie algebra of $\mathscr{G}$ into itself. So «$\det M$» is ill defined unless we perform some kind of regularization. Nevertheless, for the time being, we forget about this problem and operate only at a «formal» level. For reason which have to do with the Feynman integration, we want to write:

$$\det M = \int D\, c\, D\, \bar{c}\, e^{-\bar{C}MC} \tag{2.1.2}$$

where c and $\bar{c}$ are anticommuting scalar fields called ghost and antighost respectively.

In order to understand this formula, let us consider the equation

$$(\det A)^{-1} = \int \mathrm{d}z\, \mathrm{d}z^*\, e^{-Z^* A Z}$$

($z \in \mathbb{C}^n$, A is a complex $n \times n$ matrix) and «extend» its validity first to fields and then to anticommuting fields. The anticommutativity is responsible for the fact that we have $\det M$ instead of $(\det M)^{-1}$ in (2.1.2), according to the Berezin rules for integration over anticommuting variables (see e.g. [8]). At this point the integrand of the Feynman integrals of the theory can be written in the form $e^{-S'(A)}$, where $S'(A)$ is called the «effective» lagrangian and contains $S(A)$, $\bar{c}Mc$ and another term called gauge breaking term.

This effective lagrangian, which is the relevant object for perturbative calculations, is not any more gauge invariant, it is BRS invariant, i.e. invariant under a special class of transformations which involves also the ghost fields.

We imagine at this point the brave mathematician completely lost in the midst of these exotic, ill defined objects. Even though we cannot dig more into the quantum physics of gauge theories and we must refer to the physics literature for a complete explanation, we would like to answer in this the following questions:

a) is the space $\mathscr{C}/\mathscr{G}$ of gauge orbits a nice mathematical object?

b) does it exist a section $\sigma : \mathscr{C}/\mathscr{G} \to \mathscr{C}$?

c) what is the mathematical meaning of the Faddeev-Popov determinant?

d) what is the mathematical meaning of the ghost fields and the BRS transformations?

2.2. The space of gauge orbits in Yang-Mills theories

In order to have a nice space of gauge orbits we must, first of all, have a free action.

Now the action (1.4.1) is generally not free. Consider for instance a principal G-bundle with $G = SU(N)$ and let A be a connection, $g \in \mathcal{G}$ and $A \cdot g$ be the connection obtained through the action (1.4.1). In local coordinates we have:

$$(A \cdot g)(x) = g^{-1}(x)\, A(x)\, g(x) + g^{-1}(x)\, \mathrm{d}g(x)$$

i.e. $(A \cdot g) = A + g^{-1} \mathrm{d}_A g$.

Here A and g are written in matrix form and d_A (the covariant derivative) is locally $\mathrm{d}_A = \mathrm{d} + [A, \ \ ]$.

Then the elements $g \in \mathcal{G}$ which leave A fixed are the solutions of the equation

$$\mathrm{d}_A g = 0. \tag{2.2.1}$$

We can think to eq. (2.2.1) as a «coordinate free» equation, if we interpret the elements of $\mathcal{G}$ as sections of the bundle Hom (E^N, E^N) which are special unitary at each point; here E^N is the N-dimensional vector bundle associated to the «identity» representation of $SU(N)$ on C^N (see [9]).

Now the solutions of the equation (2.2.1) are determined by the value of g at a fixed point $x \in M$.

Infact $\forall y \in M$ consider a path ℓ connecting x and y and its horizontal lift ℓ' at $g(x) \in$ Hom (E^N, E^N); the endpoint of this path gives us an element in the fiber of Hom (E^N, E^N) over y.

So we can construct a solution for the equation (2.2.1) if and only if for any loop ℓ_0 at x the horizontal lift ℓ_0' at $g(x)$ gives us again $g(x)$, i.e. provided that

$$g(x) = a^{-1} g(x) a, \quad \forall a \in H_A = \text{holonomy group of } A. \tag{2.2.2}$$

If we want to rule out the existence of any non trivial solution to (2.2.1), in view of the (2.2.2) we have to consider $\overline{\mathcal{G}}$ instead of $\mathcal{G}$ and we have to restrict ourselves to the space $\overline{\mathcal{C}}$ of irreducible connections, i.e. the space of connections for which the only elements of G which commute with the holonomy group are the elements in the center of G. So we can state the following results

THEOREM 2.2.3. *The action of $\overline{\mathcal{G}}$ on the space of irreducible connection $\overline{\mathcal{C}}$ is free.*

Alternatively we can consider the action of $\mathcal{G}_0$ on $\mathcal{C}$ and verify that the condition $\mathrm{d}_A g = 0$ together with the condition $g(x_0) =$ identity for some $x_0 \in M$ implies $g(x) =$ identity for any x in M.

So we have:

THEOREM 2.2.4. *The action of $\mathscr{G}_0$ on $\mathscr{C}$ is free.*

We have seen before that the spaces $\mathscr{C}$, $\mathscr{G}$, $\mathscr{G}_0$ and $\overline{\mathscr{G}}$ can be considered as Hilbert manifolds (5).

We have also (see [9]):

THEOREM 2.2.5. *The space $\overline{\mathscr{C}}$ is open and dense in $\mathscr{C}$.*

This implies that also $\overline{\mathscr{C}}$ is a Hilbert manifold. The next step is to see whether the spaces $\overline{\mathscr{C}}/\overline{\mathscr{G}}$ and $\mathscr{C}/\mathscr{G}_0$ are also smooth manifolds.

We know from Bourbaki ([10]) the following:

THEOREM 2.2.6. *Let G be a Lie group (or also a group manifold), N a manifold and let $(x, g) \leadsto x \cdot g$ be a free action of G on N. Here $x \in N$ and $g \in G$. Moreover let $\mu : N \times G \to N \times N$ be a map defined by $\mu(x, g) = (x \cdot g, x)$. Then $N(N/G, G)$ is a principal fibre bundle if:*

a) *μ is a closed map*

b) *the mapping $g \leadsto x \cdot g$ is an immersion at each $x \in N$.*

Using this theorem one can prove that

$$\overline{\mathscr{C}} \longrightarrow \overline{\mathscr{C}}/\overline{\mathscr{G}} \qquad \text{and} \qquad \mathscr{C} \longrightarrow \mathscr{C}/\mathscr{G}_0$$

are principal fibre bundles, provided that the group G is one of the groups $SU(N)$ and M si one of the manifolds S^3, S^4, T^3, T^4 (see [3], [11], [12]).

We denote by $\mathscr{O}$ the manifold $\overline{\mathscr{C}}/\overline{\mathscr{G}}$ and by $\mathscr{O}_0$ the manifold $\mathscr{C}/\mathscr{G}_0$. These manifolds $\mathscr{O}$ and $\mathscr{O}_0$ are also called «spaces of gauge orbits».

We want now to find out whether the principal bundles $\overline{\mathscr{C}}(\mathscr{O}, \overline{\mathscr{G}})$ and $\mathscr{C}(\mathscr{O}_0, \mathscr{G}_0)$ are trivial or not, that is whether there exist (global) sections $\sigma : \mathscr{O} \to \overline{\mathscr{C}}$ and $\sigma_0 : \mathscr{O}_0 \to \mathscr{C}$.

It turns out that these bundles are not trivial when M is one of the manifolds S^3, S^4, T^3, T^4, and G is one the $SU(n)$s. In these cases the answer to question (b) at the end of sect. 2.1 is negative and this fact is referred to by physicists as the *Gribov ambiguity*.

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(5) In the future we will quite often omit the Sobolev index in order not to have a cumbersome notation. Nevertheless we will assume that our spaces are the proper Sobolev completions of the considered spaces of sections, unless otherwise stated.

The proof of the Gribov ambiguity is now sketched for the bundle $\overline{\mathscr{C}}(\mathcal{O},\overline{\mathscr{G}})$ when $M = S^4$ (see [9]). The first step is to prove that the manifold $\overline{\mathscr{C}}$ is a contractible space and we refer to Singer (theorem 2 in [9]) for the proof.

If $\overline{\mathscr{C}}$ is contactible and $\overline{\mathscr{C}} = \mathcal{O} \times \overline{\mathscr{G}}$ then $0 = \pi_j(\overline{\mathscr{C}}) = \pi_j(\mathcal{O}) \times \pi_j(\overline{\mathscr{G}})$ and hence $\pi_j(\overline{\mathscr{G}}) = 0$. So in order to prove the Gribov ambiguity it is enough to prove that for some j, $\pi_j(\overline{\mathscr{G}}) \neq 0$.

A direct calculation shows that

$$\pi_j(\mathscr{G}_0) = \pi_{j+4}(SU(n)) \quad ([9], \text{theorem } 5) \tag{2.2.7}$$

and from the following two exact sequences (fibrations)

$$0 \longrightarrow \mathscr{G}_0 \longrightarrow \mathscr{G} \longrightarrow SU(n) \longrightarrow 0 \tag{2.2.8}$$

$$0 \longrightarrow \mathbb{Z}_n \longrightarrow \mathscr{G} \longrightarrow \overline{\mathscr{G}} \longrightarrow 0 \tag{2.2.9}$$

we have the exact sequences of homotypy groups

$$\ldots \longrightarrow \pi_j(\mathscr{G}_0) \longrightarrow \pi_j(\mathscr{G}) \longrightarrow \pi_j(SU(n)) \longrightarrow \pi_{j-1}(\mathscr{G}_0) \longrightarrow \ldots \tag{2.2.8'}$$

$$\ldots \longrightarrow \pi_j(\mathbb{Z}_n) \longrightarrow \pi_j(\mathscr{G}) \longrightarrow \pi_j(\overline{\mathscr{G}}) \longrightarrow \pi_{j-1}(\mathbb{Z}_n) \longrightarrow \ldots . \tag{2.2.9'}$$

From (2.2.8') and from the fact that $\pi_j(SU(n)) = 0$ for $j = 0, 1, 2$ we deduce that:

$$\pi_k(\mathscr{G}) \simeq \pi_k(\mathscr{G}_0) = \pi_{k+4}(SU(n)), \text{ for } k = 0, 1.$$

From (2.2.9') it follows that the sequence

$$\pi_1(\overline{\mathscr{G}}) \longrightarrow \mathbb{Z}_n \longrightarrow \pi_0(\mathscr{G}) = 0$$

is exact and so: $\pi_0(\mathscr{G}) = 0 \Rightarrow \pi_1(\overline{\mathscr{G}}) \neq 0$.

Hence,if the group G is $SU(n)$, $n > 2$

$$\pi_0(\mathscr{G}) = \pi_4(SU(n)) = 0 \;\underline{\Rightarrow}\; \pi_1(\overline{\mathscr{G}}) \neq 0.$$

For $G = SU(2)$ we have:

$$\pi_j(\mathscr{G}) = \pi_j(\overline{\mathscr{G}}), \qquad j > 1 \tag{2.2.10}$$

due to the sequence (2.2.9), and

$$\pi_3(\mathscr{G}) \longrightarrow \pi_3(SU(2)) = \mathbb{Z} \longrightarrow \pi_2(\mathscr{G}_0) = \pi_6(SU(2)) = \mathbb{Z}_2 \rightarrow \pi_2(\mathscr{G}) \tag{2.2.11}$$

due to the sequence (2.2.8').

Hence if $\pi_2(\mathscr{G}) = \pi_3(\mathscr{G}) = 0$ then

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_{12} \longrightarrow 0 \Rightarrow Z = Z_{12} \quad \text{which is absurd.}$$

We have then sketched the proof of

THEOREM (2.2.12). *The bundle $\overline{\mathscr{C}}(\mathcal{O}, \overline{\mathscr{G}})$ is not trivial when $M = S^4$ and $G = SU(n)$.*

One can also prove that the same statement is true for $M = S^3$, T^3, T^4 and $G = SU(n)$ (see [9], [12]).

As far as the bundle $\mathscr{C}(\mathcal{O}_0, \mathscr{G}_0)$ is concerned we have the following:

THEOREM (2.2.13). *The bundle $\mathscr{C}(\mathcal{O}_0, \mathscr{G}_0)$ is not trivial when $M = S^4$ or $M = S^3$ and $G = SU(n)$.*

Proof. The space $\mathscr{C}$ is an affine space so it is obviously contractible. To prove the theorem is enough to prove that $\pi_j(\mathscr{G}_0) \neq 0$ for some j. For $M = S^4$ we have $\pi_j(\mathscr{G}_0) = \pi_{j+4}(SU(n))$ and so for $n = 2$ $\pi_2(\mathscr{G}_0)$ is equal to $\pi_6(SU(2)) = \mathbb{Z}_{12}$ and for $n > 2$ $\pi_1(\mathscr{G}_0) = \pi_5(SU(n)) = Z$. As far as the case $M = S^3$ is concerned one can prove ([9]) that $\pi_0(\mathscr{G}_0) = \pi_3((SU(n)) = Z$.

2.3. The space of Diff-orbits in gauge theories of gravitation

In gauge theories of gravitation, one works with the linear frame bundle $L(M)$ on the space-time M, and no vertical automorphisms are allowed (see the example 1.1.6). This implies that, unlike in Yang-Mills gauge theories, one can give a «dynamical status» in the theory to a family of sections of $L(M)$ $\sigma_\alpha : \mathscr{U}_\alpha \to L(M)$, locally defined on a covering $\{\mathscr{U}_\alpha\}$ of M. Indeed in a gauge theory of gravitation one generally works with two set of fields

i) a vierbein, i.e. the pull back $\sigma_\alpha^* \theta$ of the soldering form θ,

ii) a connection A on $L(M)$, locally represented by a family $\Gamma_\alpha = \sigma_\alpha^* A$ of «vector potentials» on M.

Since θ is fixed once for all, the dynamics of the vierbein depends on the sections σ_α. In most theories, these are choesen to yield a reduction of $L(M)$ to an orthonormal frame bundle on M, i.e. they give a «metric structure». In these cases we have different choices of the σ_α's giving rise to the same reduction. In physics this freedom in called the «Lorentz invariance» of the gravitational field, because two systems of sections σ_α and σ'_α give rise to the same reduction if and anly if $\sigma'_\alpha(x) = \Lambda_\alpha(x)\sigma_\alpha(x)$ ($x \in \mathscr{U}_\alpha$), where $\Lambda_\alpha(x)$ is a space-time dependent Lorentz transformation. Notice that such an invariance does not come from any transformation on $L(M)$.

In the following we shall neglect such an invariance, and we will identify all the vierbeins giving rise to the same metric field. Accordingly we will study the action of the diffeomorphism group on the space of metrics and on the space of connections. For instance in the case of the Einstein theory, we can limit our attention to the action of diffeomorphisms group on the space of metrics, while in the Hilbert-Palatini approach to General Relativity (see e.g. [13]) we have to consider the action of Diff M on the connections as well. On the other side if we consider the Eddington lagrangian we are led to study the action of Diff M on the space of connections only.

Incidentally we notice that Eddington's theory has been proved to be classically equivalent to General Relativity, via a suitable Legendre transformation ([14]). This classical equivalence suggests some interesting considerations at the quantum level (see [15]).

To be definite, in the following we will restrict our attention to the «Euclidean» version of gravity and our metrics will be properly Riemannian. To get a proper mathematical set up, we will further assume that the basis manifold M is compact, without boundary and endowed with a fixed differentiable structure.

In order to give a Hilbert structure to the manifolds of fields, we proceed as follows. We denote by $H^k(T_{(2)})$, $H^k(T_2^{\,1})$ respectively the Hilbert spaces of symmetric tensor fields of type $(0, 2)$ and of tensor fields of type $(1, 2)$ of Sobolev class k, and introduce the manifolds

$$\mathscr{M}_k = \{ \mu \mid \mu \in H^k(T_{(2)}),\ \mu \text{ is positive definite}\}$$

$$\mathscr{C}_k = \{\Gamma \mid \Gamma - \Gamma' \in H^k(T_2^{\,1})\}$$

$\mathscr{M}_k$ is an open cone in $H^k(T_{(2)})$, and thus is a Hilbert manifold (see ([4])) while $\mathscr{C}_k$ is an affine manifold modelled on $H^k(T_2^{\,1})$. As usual we assume $k > 1 +$ $+ \dim M/2$. We have now a map

$$\mathscr{S} : \mathscr{C}_{k+1} \longrightarrow H^k(T_{(2)})$$

obtained by associating to every connection its symmetrized Ricci tensor. $\mathscr{S}$ is actually a quasi-linear first order differential operator, and we learn from Palais ([5]) that it is a smooth map. Since $\mathscr{M}_k$ is open in $H^k(T_{(2)})$, then the set

$$\hat{\mathscr{C}}_{k+1} = \{\Gamma \in \mathscr{C}_{k+1} \mid \mathscr{S}(\Gamma) \in \mathscr{M}_k\}$$

is as well a Hilbert manifold, whose points are connections with invertible Ricci tensor.

We now study the actions

$$\mathscr{D}_{k+1} \times \mathscr{M}_k \longrightarrow \mathscr{M}_k$$

$$\mathscr{D}_{k+2} \times \mathscr{C}_k \longrightarrow \mathscr{C}_k$$

obtained by extending the actions of C^1 diffeomorphisms on smooth metrics and of C^2-diffeomorphisms on smooth connections. Both these actions are not free; fixed points come from metrics having isometries and from connections admitting affine transformations.

One way of getting free action is to restrict our attention to diffeomorphisms which strongly fix a point $p \in M$, i.e. to the group

$$\mathscr{D}_k^0 = :\{\psi \mid \psi \in \mathscr{D}_k \text{ s.t. } \psi(p) = p \text{ and } \psi_*|_p = \mathrm{id} \in T_pM\}$$

which is a closed subgroup of $\mathscr{D}_k$ ([16]). In [16] it is proved that $\mathscr{D}_{k+1}^0$ acts freely on $\mathscr{M}_k$ and this proof can be straightforwardly generalized show that also $\mathscr{D}_{k+2}^0$ acts freely on $\mathscr{C}_k$. Thus we get surjective maps:

$$\mathscr{M}_k \longrightarrow \mathscr{M}_k/\mathscr{D}_{k+1}^0$$

$$\mathscr{C}_k \longrightarrow \mathscr{C}_k/\mathscr{D}_{k+2}^0$$

which are continuous, provided that we give the quotient spaces the quotient topology. What it is not obvious is that the quotient spaces can be given the structure of smooth Hilbert manifolds, and that the surjective maps above define smooth fibre bundles.

The map $\mathscr{M}_k \to \mathscr{M}_k/\mathscr{D}_{k+1}^0$ has been extensively studied in the literature (see e.g. Ebin ([4]), Bourguignon ([16]) and also Marsden ([17]) for applications to General Relativity). Here we prove the following

THEOREM 2.3.1. *$\hat{\mathscr{C}}_k \to \hat{\mathscr{C}}_k/\mathscr{D}_{k+2}^0$ is a smooth fibre bundle.*

Proof. We check that the hypotheses of Theorem 2.2.6 are satisfied.
i) The action of $\mathscr{D}_{k+2}^0$ on $\hat{\mathscr{C}}_k$ is free by construction.
ii) Consider the map $\mu : \mathscr{D}_{k+2}^0 \times \hat{\mathscr{C}}_k \to \hat{\mathscr{C}}_k \times \hat{\mathscr{C}}_k$ given by $\mu(\psi, \Gamma) = (\psi^*\Gamma, \Gamma)$, where $\psi^*\Gamma$ denotes the image of Γ under the diffeomorphism ψ. Now μ is closed if and only if μ^{-1} is continuous. We want now to show that, given a converging sequence $(\psi_n^*\Gamma_n, \Gamma_n) \in \mathrm{Im}\,\mu$, then the sequence $\mu^{-1}(\psi_n^*\Gamma_n, \Gamma_n) = (\psi_n, \Gamma_n)$ converges in $\mathscr{D}_{k+2}^0 \times \hat{\mathscr{C}}_k$. Let $(\psi_n^*\Gamma_n, \Gamma_n) \to (\psi^*\Gamma, \Gamma)$; it is then enough to show that $\psi_n \to \psi$ on $\mathscr{D}_{k+2}^0$. Suppose the contrary is true, then the sequence $\mathscr{S}(\psi_n^*\Gamma_n)$ will converge to $\mathscr{S}(\psi^*\Gamma)$, because $\mathscr{S}$ is continuous, but ψ_n will not converge, and this is impossible since the action of $\mathscr{D}_{k+1}^0$ on the space of metrics is proper (see Ebin ([4])).
iii) Next we have to show that the map $\nu_\Gamma : \mathscr{D}_{k+2}^0 \to \hat{\mathscr{C}}_k$ given by $\nu_\Gamma(\psi) = \psi^*\Gamma$ is an injective immersion for any $\Gamma \in \hat{\mathscr{C}}_k$. Now, ν_Γ is obviously injective, since the action is free. It is also closed, because a convergent sequence $\psi_n^*\Gamma$ on an orbit, yields a convergent sequence ψ_n on $\mathscr{D}_{k+2}^0$ by the same argument used in (ii). The tangent map $T\nu_\Gamma : T_e\mathscr{D}_{k+2}^0 \to T_\Gamma\hat{\mathscr{C}}_k$ is obviously injective, since it is given by the Lie derivative $T\nu_\Gamma(X) = \mathscr{L}_X\Gamma$, which vanishes iff $X = 0$. Moreover it is a second order differential operator with injective symbol. Accordingly $T_\Gamma\hat{\mathscr{C}}_k$ splits, by the usual argument of the Fredholm alternative.

Remark. From the proof above, it is clear the reason why we restricted our attention to the manifold $\hat{\mathscr{C}}_k$, instead of working on the whole $\mathscr{C}_k$. In this way we could infact benefit from the results already available for the space of metrics, thanks to the fact that $\mathscr{S}(\ )$ is continous.

As a final remark, we notice that there exists a map $\bar{\mathscr{S}}$ such that the diagram

$$\begin{tikzcd} \hat{\mathscr{C}}_{k+1} \arrow[r, "\mathscr{S}"] \arrow[d] & \mathscr{M}_k \arrow[d] \\ \hat{\mathscr{C}}_{k+1}/\mathscr{D}^0_{k+3} \arrow[r, "\bar{\mathscr{S}}"] & \mathscr{M}_k/\mathscr{D}^0_{k+1} \end{tikzcd}$$

commutes. Hence the Ricci map $\mathscr{S}$ is a smooth bundle homomorphism.

3. A CONNECTION AND A RIEMANNIAN METRIC FOR THE SPACE OF GAUGE ORBITS

3.1. Coordinate system on the space of gauge orbits

We want now to describe more closely the structure of space $\mathscr{O}$.

The first step is to be define some operations in $\Omega^*(M, a\mathrm{d}P)$:

a) the Hodge duality operator:

$$(3.1.1) \qquad * : \Omega^p(M, a\mathrm{d}P) \longrightarrow \Omega^{\dim M - p}(M, a\mathrm{d}P)$$

which is the obvious extension of the Hodge operator for ordinary forms on M.

b) the cup operator

$$(3.1.2) \qquad \Lambda : \Omega^p(M, a\mathrm{d}P) \times \Omega^q(M, a\mathrm{d}P) \longrightarrow \Omega^{p+q}(M)$$

which combines the exterior product for ordinary forms on M with the invariant inner product on G' (Killing form).

c) inner product

$$(3.1.3) \qquad (\ |\ ) : \Omega^p(M, a\mathrm{d}P) \times \Omega^p(M, a\mathrm{d}P) \longrightarrow \mathbb{R}$$

$$(\psi \mid \omega) =: \int_M \psi \Lambda * \omega.$$

For any $A \in \mathscr{C}$, let d_A denote, as usual, the covariant derivative, i.e.

$$\mathrm{d}_A : \Omega_k^*(M, \mathrm{ad}P) \longrightarrow \Omega_{k-1}^{*+1}(M, \mathrm{ad}P) \qquad (6)$$

The operator d_A^* will be, by definition, the adjoint of d_A with respect to the inner product (3.1.3). Obviously

$$\mathrm{d}_A^* : \Omega_k^*(M, \mathrm{ad}P) \longrightarrow \Omega_{k-1}^{*-1}(M, \mathrm{ad}P)$$

and both d_A and d_A^* are linear operators which are continuous in the Sobolev norms.

We have then the following:

THEOREM (3.1.4). *$\forall A \in \overline{\mathscr{C}}_k$ the operator $\mathrm{d}_A^* \mathrm{d}_A : \mathscr{G}'_{k+1} \to \mathscr{G}'_{k-1}$ is a continuous bijection.*

Proof. See [11].

We have also:

THEOREM (3.1.5). *For any $A \in \overline{\mathscr{C}}_k$, $\mathrm{d}_A(\mathscr{G}'_{k+1})$ and $\ker \mathrm{d}_A^* \mid \Omega_k^1(M, \mathrm{ad}P)$ are two closed subspaces which are orthogonal w.r.t. the inner product (3.1.3). We have also $\Omega_k^1(M, \mathrm{ad}P) = \mathrm{d}_A(\mathscr{G}'_{k+1}) \oplus \ker \mathrm{d}_A^*$.*

Proof. We have

$$\mathrm{d}_A(\mathscr{G}'_{k+1}) = \ker (1 - \mathrm{d}_A(\mathrm{d}_A^* \mathrm{d}_A)^{-1} \mathrm{d}_A^*)$$

hence both $\mathrm{d}_A(\mathscr{G}'_{k+1})$ and $\ker \mathrm{d}_A^*$ are closed since they are the kernel of closed operators in a Hilbert space. Moreover $\forall\, \psi \in \ker \mathrm{d}_A^*$, $\tau \in \Omega_k^0(M, a\mathrm{d}P)$ we have: $(\psi \mid \mathrm{d}_A \tau) = (\mathrm{d}_A^* \psi \mid \tau) = 0$.

Finally $\forall \varphi \in \Omega_k^1(M, a\mathrm{d}P)$, we have:

$$\varphi = \mathrm{d}_A(\mathrm{d}_A^* \mathrm{d}_A)^{-1} \mathrm{d}_A^* \varphi + (\varphi - \mathrm{d}_A(\mathrm{d}_A^* \mathrm{d}_A)^{-1} \mathrm{d}_A^* \varphi)$$

where the second term obviously belongs to $\ker \mathrm{d}_A^*$.

We denote now $\mathrm{d}_A(\mathscr{G}'_{k+1})$ by V_A and $\ker \mathrm{d}_A^*$ by H_A. We are then able to split each tangent space at $A \in \overline{\mathscr{C}}_k$ into two supplementary subspaces H_A and V_A; V_A is, by definition, the vertical subspace at $A \in \overline{\mathscr{C}}_k$ as it is evident from the study of the case (i) in Sect. 1.4, Moreover due to the $\mathscr{G}$-invariance of the

(6) In this section, for the sake of clarity, we will resume the Sobolev indices.

inner product (3.1.3), the space H_A, which is the orthogonal complement of V_A, satisfies the condition:

$$H_{A \cdot g} = (Rg)^* H_A$$

here Rg denotes the right action by $g \in \mathcal{G}$. Finally since H_A depends smoothly on A, we have

THEOREM 3.1.6. *The subspaces H_A, $A \in \overline{\mathcal{C}}_k$ define a connection on $\overline{\mathcal{C}}_k(\mathcal{O}_k, \overline{\mathcal{G}}_{k+1})$.*

Consider now the following subsets of $\overline{\mathcal{C}}_k$:

$$\mathcal{F}_k(A) = \{A' \in \overline{\mathcal{C}}_k \mid \mathrm{d}_A^*(A' - A) = 0\}$$

$$\mathcal{S}_k^\delta(A) = \{A' \in \overline{\mathcal{C}}_k \mid \|A - A'\| < \delta\}.$$

Then we have:

THEOREM 3.1.7.

$$\forall A \in \overline{\mathcal{C}}_k \ \ \exists \delta > 0 \ \ s.t. \ \ Q_k^\delta(A) \equiv \mathcal{F}_k(A) \cap \mathcal{S}_k^\delta(A)$$

does not contain two gauge equivalent connections.

Proof. See [18].

Let π now denote the projection map in the bundle $\overline{\mathcal{C}}_k(\mathcal{O}_k, \overline{\mathcal{G}}_k)$. Due to theorem 3.1.7, π maps homeomorphically $Q_k^\delta(A)$ onto an open subset $U_A \equiv \pi(Q_k^\delta(A))$ of $\mathcal{O}_k$. If we denote by π_A the projection map restricted to $Q_k^\delta(A)$ and by ρ_A the map which assign to $A' \in \mathcal{F}_k(A)$ the element of $\Omega_k^1(M, a\mathrm{d}P)$ given by $A' - A$, then an atlas for $\mathcal{O}$ is given by

$$\{U_A, \rho_A \circ \pi_A^{-1}, H_A\}, \qquad A \in \overline{\mathcal{C}}_k$$

(see [3]).

Notice that H_A and $H_{\tilde{A}}$ for different A and $\tilde{A}$ are isomorphic as Hilbert space.

We would like now to write the expression of the connection form and of the curvature form corresponding to the connection given in theorem 3.1.6.

THEOREM 3.1.8. *The connection form and the curvature form of the connection given by the splitting $T_A \overline{\mathcal{C}}_k = H_A \oplus V_A$ are as follows:*

a) connection form α

$$\alpha_A : T_A \overline{\mathcal{C}}_k \longrightarrow \Omega_k^0(M, a\mathrm{d}P) \simeq \mathcal{G}_k'$$

$$\tau \rightsquigarrow (\mathrm{d}_A^* \mathrm{d}_A)^{-1} \mathrm{d}_A^* \tau$$

b) curvature form K

$$k_A : T_A\overline{\mathscr{C}}_k \times T_A\overline{\mathscr{C}}_k \longrightarrow \Omega^0_k(M, \mathrm{ad}P) \simeq \mathscr{G}'_k$$
$$\tau_1, \tau_2 \rightsquigarrow (\mathrm{d}^*_A \mathrm{d}_A)^{-1}(b^*_{\tau_1^h}(\tau_2^h) - b^*_{\tau_2^h}(\tau_1^h))$$

where τ_i^h, $i = 1, 2$ *is the horizontal component of* τ_i, $b_{\tau_i^h}$ *is the operator* $b_{\tau_i^h} = [\tau_i^h, .\,]$ *and* $b^*_{\tau_i^h}$ *is the adjoint of* $b_{\tau_i^h}$ *w.r.t. to the inner product* (3.1.3).

Proof. (a) is obvious from the proof of theorem 3.1.5; as far as (b) is concerned see Singer ([9]).

As a final remark, notice that the choice of the connection as in theorem (3.1.6) corresponds in the physicists language to the choice of the «covariant background gauge».

3.2. A weak Riemannian metric on the space of gauge orbits

In this section we want to construct a «weak» Riemannian metrics on the space of gauge orbits. Later on we will establish a relation between this metric and the Faddeev-Popov determinant. By a (weak) Riemannian metric in a Hilbert Manifold, we mean the assignment in a smooth way of a non degenerate inner product in each tangent spaces.

A Riemannian metric is called «strong» if the induced metric in each tangent space is equivalent to the one induced by the original inner product. We define now $\forall X_1, X_2 \in T_a\mathcal{O}, a' \in \mathcal{O}$:

(3.2.1) $$\tilde{g}_{a'}(X_1, X_2) = (X_1^h, X_2^h)_{A'}$$

where $A' \in \pi^{-1}(a')$; $(\ ,\ )_{A'}$ is the inner product in $H_{A'}$ induced by the inner product (3.1.3) and X_i^h $(i = 1, 2)$ is the horizontal lift of X_i $(i = 1, 2)$ w.r.t. to the connection given by theorem (3.1.6).

Let us now choose $a \in \mathcal{O}$, $A \in \pi^{-1}(a)$, $a' \in \pi\, Q_k^\delta(A)$ and $A' \in Q_k^\delta(A)$ with $\pi(A') = a'$. In the local coordinates given in sect. 3.1 we have:

(3.2.2) $$\tilde{g}_{a'}(X_1, X_2) = (\tau_1 \,|\, q(A')\, \tau_2)$$

where $\tau_i \in T_{A'}(Q_k^\delta(A)) = H_A$, $q(A') = 1 - \mathrm{d}_{A'}(\mathrm{d}^*_A \mathrm{d}_{A'})^{-1}\mathrm{d}^*_{A'}$ and τ_i satisfies the equation $X_i^h = q(A')\,\tau_i$.

Since $q(A')$ is the projection operator onto $H_{A'}$ it is evident that the expression (3.2.2) coincides with (3.2.1) for elements $a' \in \mathcal{O}$ in the neighborhood $\pi\, Q_k^\delta(A)$ of a.

The formula (3.2.2) does not give any more a good expression for the metric (3.2.1) when the operator $q(A')$ restricted to H_A is not invertible, that is when part of the orbit through a' is tangent to $\mathscr{F}_k(A)$; i.e. when the «Gribov ambiguity» appears.

The metric (3.2.1) gives a norm in each tangent space $T_a(\mathcal{O})$, $a \in \mathcal{O}$ which is weaker than the Sobolev norm, that is (3.2.1) gives a weak Riemannian structure.

3.3. Faddeev-Popov determinant

Let now A be a connection and A^{loc} be its expression in local coordinates. The «vector potential» A^{loc} is then a G'-valued 1-form on M.

According to the physical literature a gauge condition is a condition of the form

$$f(A^{\mathrm{loc}}) = 0 \tag{3.3.1}$$

where f maps (locally defined) G'-valued 1-forms on M into G'-valued 0-forms on M. Let g^{loc} be the local expression of the gauge transformation g and consider the mapping:

$$g^{\mathrm{loc}} \leadsto f[(g^{\mathrm{loc}})^{-1} A^{\mathrm{loc}} g^{\mathrm{loc}} + (g^{\mathrm{loc}})^{-1} \mathrm{d} g^{\mathrm{loc}}]. \tag{3.3.2}$$

The Faddeev-Popov operator $M_f(A^{\mathrm{loc}})$ is defined as the functional derivative of the mapping (3.3.2) at $g^{\mathrm{loc}} = $ identity.

$M_f(A^{\mathrm{loc}})$ is an operator which maps the space of G'-valued 0-forms on M into itself. So, in a very much «heuristic» sense, $M_f(A^{\mathrm{loc}})$ is a square matrix and one can consider its determinant, which is, by definition the Faddeev-Popov determinant $\Delta^{\mathrm{F.P.}}$.

The so called «background gauge condition» is particularly simple and, among other things, has the advantage that it can be written down, without resorting to the local expression of the connection. The background gauge condition reads:

$$\mathrm{d}^*_{A_0}(A - A_0) = 0 \tag{8.3.3}$$

where A_0 is a fixed connection. In this case the operator $M_f(A) = M_{A_0}(A)$ is the operator $\mathrm{d}^*_{A_0}\mathrm{d}_A$.

Babelon and Viallet ([19]) showed that for the backgorund gauge, the Faddeev--Popov determinant is related in a formal sense to the metric (3.2.1). More exactly they write:

$$(\det \tilde{g})^{1/2} = \frac{\det(\mathrm{d}^*_{A_0}\mathrm{d}_A)}{[\det(\mathrm{d}^*_{A_0}\mathrm{d}_{A_0})]^{1/2}[\det(\mathrm{d}^*_A\mathrm{d}_A)]^{1/2}} . \tag{3.3.4}$$

The equation (3.3.4) is valid, of course, only in a formal sense, since the operators involved are operators on infinite dimensional Hilbert space not necessarily of the trace class, so their determinants are ill defined (i.e. infinite) unless some regularizing procedure is performed. The functional Integral for gauge theories is defined as

$$I = \int_{e} DA\, e^{-S(A)} \tag{3.3.5}$$

where $S(A)$ is the action and DA is a formal («Lebesgue») measure on the space of connections.

The integral (3.3.5) can be also written as

$$I = N \int_{e} DA\; \delta[\mathrm{d}^{*}_{A_0}(A - A_0)]\, \Delta^{\mathrm{F.P.}} e^{-S(A)}. \tag{3.3.6}$$

The expression (3.3.6) is the starting point for perturbative calculations in quantum gauge theories.

Taking into account (3.3.4) one can also write:

$$I = N \int_{\mathcal{C}} Da\, (\det \tilde{g})^{1/2} (\det\, (\mathrm{d}^{*}_{A_a} \mathrm{d}_{A_a})^{1/2} e^{-S(a)} \tag{3.3.7}$$

where Da is the formal («Lebesgue») measure on the space of gauge orbits, $\det \tilde{g}$ is the formal determinant of the metric (3.2.1), $S(a)$ is the (gauge invariant) action, A is any connection such that $\pi(A) = a$. Formula (3.3.6) tell us that the volume element relative to the metric (3.2.1) gives the right measure for the functional integral of the theory.

Unfortunately everything which has been said in this section is only true at a formal level; a mathematically correct definition for the functional integral is yet to be found. Nevertheless the theory and the relative perturbative calculations give us some real physical results and predictions.

4. GHOSTS AND ANOMALIES

4.1. Ghosts and BRS transformations

In order to be able to make perturbative calculations one has to transform the integral (3.3.6) into another functional integral, whose integrand is of the form «$\exp\{-S'\}$» as it was mentioned in sect. 2.1.

The functional S' is called the effective action and it turns out to contain the action $S(A)$ plus a term which comes out of the «Fourier transform» of the δ-function in (3.3.6) (gauge breaking term) plus the ghost term which arises in the following way.

Consider first the equation

$$(4.1.1) \qquad (\det A)^{-1} = \int dz\, dz^* \exp(-z^* A z)$$

where A is an $n \times n$ complex matrix, and the integral is computed over $C^n \times C^n$.

If we pretend, as it was already said in sect. 2.1, that the validity of (4.1.1) extends to functional integrals and if we assume that the Berezin's rules for integration over anticommuting variables (see e.g. [8]) hold, then we have:

$$(4.1.2) \qquad \Delta^{\text{F.P.}} = \det M_f = \int dc\, d\bar{c} \exp(-\bar{c} M_f c)$$

where c and $\bar{c}$ are anticommutative G'-valued fields called ghost and antighost respectively. Notice that the anticommutativity is responsible for the fact that in the r.h.s. of (4.1.2) we have $\det M_f$ instead of $(\det M_f)^{-1}$ as it should be in analogy with the (4.1.1).

Now the effective action S' depends not only on the vector potential and the matter fields, but also on the ghost and antighost fields.

Since we deal essentially with pure gauge theories, we will not consider matter fields in the future.

The effective action is not more gauge invariant, it is B.R.S. invariant i.e. it is invariant under the Becchi-Rouet-Stora (BRS) transformations whose «infinitesimal» version is

$$(4.1.3.a) \qquad A^{\text{loc}} \longmapsto A^{\text{loc}} + \lambda d_A c = : A^{\text{loc}} + \lambda s(A^{\text{loc}})$$

$$(4.1.3.b) \qquad c \longmapsto c - (1/2)\lambda [c, c] = : c + \lambda s(c)$$

$$(4.1.3.c) \qquad \bar{c} \longmapsto \bar{c} + \lambda B = : \bar{c} + \lambda s(\bar{c})$$

where A^{loc} is the vector potential (connection), c and $\bar{c}$ are the ghost and the antighost fields, s is, by definition, the (infinitesimal) BRS operator, B is a suitable auxiliary fields such that $s(B) = 0$ and λ is a «constant» anticommuting parameter which is a sort of a conventional device which is used in field theory whenever a match between commuting and anticommuting objects is needed.

He have also, due to the anticommutativity of the ghost fields:

$$s(d_A c) = -(1/2)\, d_A([c, c] + [d_A c, c]) = 0.$$

So $s^2 = 0$ (nilpotent character of the BRS operator).

Let now $h(A^{\mathrm{loc}}, c)$ be a functional of the vector potentials, of the ghost fields (7) such that $h(A^{\mathrm{loc}}, c) = \int_M h_n(A^{\mathrm{loc}}, c)$, where $h_n(A^{\mathrm{loc}}, c)$ is an n-form on M ($n = \dim M$) whose local expression is a polynomial in the fields A^{loc}, c and their derivatives. The BRS operator can be defined on these functionals in the following way:

$$(Sh)(A^{\mathrm{loc}}, c) = : \int_M \left[s(A^{\mathrm{loc}}) \frac{\delta}{\delta A^{\mathrm{loc}}} + s(c) \frac{\delta}{\delta c} \right] h_n(A^{\mathrm{loc}}, c) +$$

where $\dfrac{\delta}{\delta A}$ and $\dfrac{\delta}{\delta c}$ are functional derivatives.

An anomalous term (like the Adler-Bell-Jackiw anomaly) can be defined as a functional Q of the vector potentials and the ghosts, such that $SQ = 0$ and $\nexists Q'$ such that $Q = SQ'$ (for the definition of the Adler Bell Jackiw anomaly see [20], [21]; for the definition of anomalous terms see [22]).

We are now able to explain the geometrical meaning of the ghost field and the BRS transformations.

Consider the $\mathcal{G}'$-valued left invariant 1-form ω on $\mathcal{G}$ (Maurer Cartan form) defined by:

$$\omega(\xi) = \xi, \qquad \forall \xi \in \mathcal{G}'$$

(we identify the left invariant vector fields on with the corresponding elements of $\mathcal{G}'$). We identify ω with the ghost field (for more details see [23]).

For any connection A and $\forall \xi \in \mathcal{G}'$, $A + \mathrm{d}_A(\omega(\xi))$ is another connection, which is obtained form A, by performing the infinitesimal gauge transformation given by ξ. If ω is the ghost field, the transformation (4.1.3.a) becomes:

$$A \longmapsto A + \mathrm{d}_A \omega$$

which in turn can be seen as representing in a symbolic form the effect of all the possible infinitesimal gauge transformations. In other words (4.1.3.a) stands for:

$$(4.1.3.\mathrm{a}') \qquad (A, \xi) \rightsquigarrow A + \mathrm{d}_A(\omega(\xi)).$$

Notice that $\mathrm{d}_A \omega$ can be considered as a $\Omega^1(M, adP)$-valued 1-form on $\mathcal{G}$ such that: $(\mathrm{d}_A \omega)(\xi) = \mathrm{d}_A(\omega(\xi))$. If we apply twice the transformation (4.1.3.a'),

(7) One should also take into account matter fields, but, as it was said before, in this paper we avoid considering them.

antisymmetrizing with respect to the vector fields ξ, $\zeta \in \mathcal{G}'$ involved, we have:

$$(A, \xi, \zeta) \rightsquigarrow A + \mathrm{d}_A\, \omega(\xi) + \mathrm{d}_{A + \mathrm{d}_A \omega(\xi)} \omega(\zeta)$$
$$- \mathrm{d}_A\, \omega(\zeta) - \mathrm{d}_{A + \mathrm{d}_A \omega(\zeta)} \omega(\xi) =$$
$$A + \mathrm{d}_A [\omega(\xi), \omega(\zeta)] = : A + [\mathrm{d}_A\, \omega, \omega]\, (\xi, \zeta).$$

Notice that $[\mathrm{d}_A\, \omega, \omega]$ is a 2-form on $\mathcal{G}$ with values in $\Omega^1(M, a\mathrm{d}P)$ which satisfies the equation:

$$[\mathrm{d}_A\, \omega, \omega] - (1/2)\, \mathrm{d}_A\, [\omega, \omega] = \tag{4.1.4}$$
$$= [\mathrm{d}_A\, \omega, \omega] + \mathrm{d}_A (\mathrm{d}_{\mathcal{G}} \omega) = 0$$

where $\mathrm{d}_{\mathcal{G}}$ is the exterior derivative for forms defined on $\mathcal{G}$.

We can then conclude ([23]):

a) we know that the ghost is an anticommuting scalar field: this is explained by the fact that it is a 1-form on $\mathcal{G}$ (anticommutativity) with values in a space of zero forms on M (scalar character);

b) the transformation (4.1.3.a) can be identified with the transformation (4.1.3.a$'$);

c) in the transformation (4.1.3.b) the operator s can be identified with the exterior derivative for forms on $\mathcal{G}$;

d) eq. (4.1.4) expresses the nilpotent character of the BRS transformations, that is, if we apply twice the transformation (4.1.4.a$'$), antisymmetrizing w.r.t. to the vector fields involved, and add the term $\mathrm{d}_A((\mathrm{d}_{\mathcal{G}} \omega)(\xi, \zeta)$, we come back to the same connection where we started from.

The study of the geometrical meaning of the BRS transformations has led us to consider that the relevant object is the action of $\mathcal{G}$ on $\mathcal{C}$. Now, whenever we have an action of a Group on a manifold, we have, in a natural way, a cohomology of the Lie algebra of the Group with coefficients in the algebra of differential forms on the manifold. In our case the coboundary operator is exactly the BRS operator, as we shall in the next section.

4.2. Cohomology of the gauge Lie algebra

Let $\Omega^0(\mathcal{C})$ be the space of zero forms (functionals) on $\mathcal{C}$. $\forall \xi \in \mathcal{G}'$ we can define the fundamental vector field Z_ξ as the vector field which acts on $\Omega^0(\mathcal{C})$ in the following way:

$$(Z_\xi f)(A) = : \frac{\mathrm{d}}{\mathrm{d}t} f(A + t\,\mathrm{d}_A\, \xi)|_{t=0}.$$

Notice that $Z_\xi(A) = \mathrm{d}_A\, \omega(\xi)$.

Let $\Gamma^p(\mathcal{G}',\mathcal{C})$ be the space of the p-linear skew mappings from $\mathcal{G}' \times \mathcal{G}' \times \ldots \times \mathcal{G}'$ (p times) to $\Omega^0(\mathcal{C})$, which satisfy the following continuity condition: if $\psi \in \Gamma^p(\mathcal{G}',\mathcal{C})$ and $A \in \mathcal{C}$, then the map ψ_A given by

$$\psi_A : \mathcal{G}' \times \mathcal{G}' \times \ldots \times \mathcal{G}' \longrightarrow \mathbb{R}$$

$$(\xi_1, \xi_2, \ldots, \xi_p) \rightsquigarrow \psi(\xi_1, \xi_2, \ldots, \xi_A)(A)$$

is continuous $\forall A \in \mathcal{C}$.

In $\Gamma^*(\mathcal{G}',\mathcal{C}) =: \oplus_p \Gamma^p(\mathcal{G}',\mathcal{C})$ we can define a coboundary operator δ by:

$$\delta : \Gamma^p(\mathcal{G}',\mathcal{C}) \longrightarrow \Gamma^{p+1}(\mathcal{G}',\mathcal{C})$$

$$(\delta\psi)(\xi_1, \ldots, \xi_{p+1}) =:$$

$$=: \Sigma_{i=1}^{p+1}(-1)^{i+1} Z_{\xi_i}(\psi(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{p+1}) +$$

$$+ \sum_{i<j} (-1)^{i+j} \psi([\xi_i, \xi_j], \xi_1, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_{p+1})$$

where $\hat{}$ indicates omission and $\xi_i \in \mathcal{G}'$, $\forall i$.

Let us denote by $H^p(\mathcal{G}',\mathcal{C})$ the p-th cohomology group for the complex $(\Gamma^*(\mathcal{G}',\mathcal{C}), \delta)$. Then we have (see [23]):

a) functionals of the vector potential with p-ghosts are elements of $\Gamma^p(\mathcal{G}',\mathcal{C})$ (p-cochains);

b) the BRS operator can be identified with the coboundary operator δ;

c) gauge invariant functionals with no ghosts are elements of $H^0(\mathcal{G}',\mathcal{C})$;

d) (integrated) anomalous terms with p-ghosts ($p \geqslant 1$) are elements of $H^p(\mathcal{G}',\mathcal{C})$.

Let now $\overline{\mathcal{C}}\,(\overline{\mathcal{C}}/\overline{\mathcal{G}}, \overline{\mathcal{G}})$ be, as usual, the principal bundle defined in sect. 2.2 endowed with the connection of theorem (3.1.6). Obviously $\overline{\mathcal{G}}' = \mathcal{G}'$ and one can consider the complex $\Gamma^*(\mathcal{G}',\overline{\mathcal{C}})$ defined in the same way as before. A p--form ϕ on $\overline{\mathcal{C}}$ ($p \geqslant 1$) is called vertical if $\phi(A, \tau_1, \ldots, \tau_p) = 0$ whenever one of $\tau_i \in T_A\overline{\mathcal{C}}$ is horizontal.

We have then the following theorems (see [23] for the proofs)

THEOREM 4.2.1. *The space of vertical p-forms ($p \geqslant 1$) on $\overline{\mathcal{C}}$ is isomorphic to $\Gamma^p(\mathcal{G}',\overline{\mathcal{C}})$.*

In particular to any $\psi \in \Gamma^p(\mathcal{G}',\overline{\mathcal{C}})$, theorem 4.2.1. associates the vertical p-form $\hat{\psi}$ on $\overline{\mathcal{C}}$ given by:

$$\hat{\psi}(A; \tau_1, \ldots, \tau_p) = \psi(\alpha_A(\tau_1), \ldots, \alpha_A(\tau_p))(A)$$

where $A \in \overline{\mathscr{C}}$, $\tau_i \in T_A \overline{\mathscr{C}}$ and α_A is the connection form as in theorem (3.1.8).

Let now ϕ be any p-form on $\overline{\mathscr{C}}$ and let ϕ_v denote the vertical p-form defined by:

$$\phi_v(A; \tau_1, \ldots, \tau_p) =:$$

$$\phi(A; \mathrm{d}_A \alpha_A(\tau_1), \ldots, \mathrm{d}_A \alpha_A(\tau_p)).$$

If we denote by $\overline{\delta}$ the exterior derivative operator for forms on $\overline{\mathscr{C}}$ and, as usual, by δ the coboundary (BRS) operator acting on cochains in $\Gamma^*(\mathscr{G}', \overline{\mathscr{C}})$, we have (see [23]):

THEOREM 4.2.2. *The following identity holds:*

$$(\delta \psi)(\zeta_1, \ldots, \zeta_{p+1}) = (\overline{\delta} \hat{\psi})_v (A; \tau_1, \ldots, \tau_{p+1})$$

where $\psi \in \Gamma^p(\mathscr{G}', \overline{\mathscr{C}})$, $\zeta_i = \alpha_A(\tau_i) \in \mathscr{G}'$, $\tau_i \in T_A \overline{\mathscr{C}}$ and $\hat{\psi}$ is the vertical p-form on $\overline{\mathscr{C}}$ defined as before.

4.3. Local cohomology and ABJ anomaly

The space $\Gamma^p(\mathscr{G}', \overline{\mathscr{C}})$ is «too large» for field theory; what we really need are functionals which are integrals of polynomials in the fields, the ghosts and their derivatives. For this reason we want to define the «local» cohomology.

Let B a generic vector bundle over the manifold M and let $\Gamma(B)$ the space C^∞ sections of B.

A p-linear map L from

$$\Gamma(B) \times \ldots \times \Gamma(B) \quad (p\text{-times}) \text{ to } \Omega^k(M)$$

is said to be local if

$$\operatorname{supp} L(\xi_1, \ldots, \xi_p) \subset \operatorname{supp} \xi_1 \cap \ldots \cap \operatorname{supp} \xi_p .$$

A theorem by Peetre [24], guarantees that in any local coordinates system L is represented by a differential operator.

Let us consider now m-linear local maps L_n from

$$\Omega^1(M, ad P) \times \ldots \times \Omega^1(M, ad P)(m\text{-times}) \text{ to } \Omega^k(M).$$

Choose $A_0 \in \mathscr{C}$, set $\eta(A) =: A - A_0$ and consider expressions of the form

$$L(A) = \sum_{n=0}^{s} L_n(\eta(A), \ldots, \eta(A))$$

where the L_n*'s* are m-linear local maps as before.

In local coordinates $L(A)$ is a k-form whose components are polynomials in the $\eta(A)$ and its derivatives.

We denote by $D^k(\mathscr{C}, A_0, M)$ the space of all such k-form which are made out of a finite number of L_n's.

We consider then p-cochains from $\mathscr{G}' \times \mathscr{G}' \times \ldots \times \mathscr{G}'$ to $L^k(\mathscr{C}, A_0, M)$ with the property that, if ϕ is such a cochain then:

$$\operatorname{supp} \phi(\zeta_1, \ldots, \zeta_p)(A) \subset \operatorname{supp} \zeta_1 \cap \ldots \cap \operatorname{supp} \zeta_p .$$

In local coordinates such a ϕ is a k-form depending «locally» on p-ghosts and on the vector potential.

We denote by $S^{p,k}$ the space of such ϕ's and set $S =: \sum_{p,k} S^{p,k}$.

The space $S^{p,k}$ is in fact a double complex since we can introduce two coboundary operators as follows:

a)
$$\mathrm{d} : S^{p,k} \longrightarrow S^{p,k+1}$$
$$(\mathrm{d}\phi_p^k)(\zeta_1, \ldots, \zeta_p) =: \mathrm{d}(\phi(\zeta_1, \ldots, \zeta_p))$$
$$\phi_p^k \in S^{p,k}, \qquad \zeta_i \in \mathscr{G}'.$$

b)
$$\delta_{\mathrm{loc}} : S^{p,k} \longrightarrow S^{p+1,k}$$
$$(\delta_{\mathrm{loc}}\phi_p^k)(\zeta_1, \ldots, \zeta_{p+1}) =:$$
$$\Sigma_{i=1}^{p}(-1)^{i+1}\Theta_{\mathrm{loc}}(\zeta_i)\, \phi_p^k(\zeta_1, \ldots, \hat{\zeta}_i, \ldots, \zeta_{p+1}) +$$
$$+ \sum_{i<j} (-1)^{i+j} \phi_p^k([\zeta_i, \zeta_j], \zeta_1, \ldots, \hat{\zeta}_i, \ldots, \hat{\zeta}_j, \ldots, \zeta_{p+1})$$

where $\hat{}$ denotes omission and $\Theta_{\mathrm{loc}}(\zeta)$ is the operator defined as follows:

$$\Theta_{\mathrm{loc}}(\zeta) : D^k(\mathscr{C}, A_0, M) \longrightarrow D^k(\mathscr{C}, A_0, M)$$
$$(\Theta_{\mathrm{loc}}(\zeta)\psi)(A) \leadsto \frac{\mathrm{d}}{\mathrm{d}t}\, \psi(A + t\,\mathrm{d}_A \zeta)\big|_{t=0} .$$

Of course $\mathrm{d}\,\delta_{\mathrm{loc}} = \delta_{\mathrm{loc}}\mathrm{d}$. We can then prove that (see [23]):

a) if $n = \dim M$ and ϕ_p^n belongs to $S^{p,n}$, then $\int_M \phi_p^n \in \Gamma^p(\mathscr{G}', \mathscr{C})$ and

$$\int_M \delta_{\mathrm{loc}} \phi_p^n = \delta \int_M \phi_p^n ;$$

b) $\int_M \phi_p^n \in H^p(\mathscr{G}',\mathscr{C}) \Rightarrow \exists\ \phi_{p+n}^0 \in S^{p+n,0}$, such that ϕ_{p+n}^0 belongs to the $(p+n)$- -th cohomology group computed w.r.t. the coboundary operator δ_{loc} (we prove more generally that the spectral sequence for the double complex S^{**} is degenerated to the E_2 term);
c) under the correspondence mentioned above, the Adler-Bell-Jackiw anomaly and the «anomaly» in two dimension (see [23]) correspond respectively to ϕ_5^0 and ϕ_3^0 given by:

$$\phi_5^0(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = f_3(\xi_1, [\xi_2, \xi_3), [\xi_4, \xi_5]), \quad \xi_i \in \mathscr{G}'$$

and

$$\phi_3^0(\xi_1, \xi_2, \xi_3) = \mathscr{K}(\xi_1, [\xi_2, \xi_3]), \qquad \xi_i \in \mathscr{G}'$$

where f_3 and K are respectively the trilinear symmetric invariant map for G' and the Killing form (we consider that every ad-invariant multilinear symmetric map can be estend to a map from

$$\Omega^{i_1}(M, a\mathrm{d}P) \times \Omega^{i_2}(M, a\mathrm{d}P) \times \ldots \text{ to } \Omega^{i_1+i_2\cdots}(M)).$$

4.4. Calculus of variations and anomalies. The trace anomaly

The cohomological interpretation of the ABJ-anomalies suggests that anomalies in field theory are in general of cohomological nature. To develop such a general interpretation we resort here to the geometrical set-up for calculus of variations as differential calculus on Banach manifolds, and we give an important example which confirms the above conjecture.

Let $\mathscr{F}$ be a Banach manifold, whose points ϕ are fields over a 4-dimensional manifold M, and let S be a smooth function on $\mathscr{F}$, playing the role of the action of a physical theory. We assume that S is local, i.e. it can be written as the integral over M of a polynomial lagrangean density. The space of such functions will be denoted by Ω_{loc}^0. For the sake of simplicity we also assume that M is compact and without boundaries.

Let now $Z \in \mathfrak{X}(\mathscr{F})$ be a vector field over $\mathscr{F}$. The first variation of S along Z is given by $\mathscr{L}_Z S = \mathrm{d}S(Z)$, where $\mathscr{L}_Z$ denotes the Lie derivative along Z and d is the exterior differential. The invariance group of the theory is a subgroup $\mathscr{K}$ of the diffeomorphism group of $\mathscr{F}$, which acting on $\mathscr{F}$ leaves the action S invariant. We imagine that $\mathscr{K}$ can be given the structure of a Lie group and we denote by $\mathscr{K}$ its Lie algebra. The action of $\mathscr{K}$ on $\mathscr{F}$ induces a Lie algebra homomorphism $\lambda : \mathscr{K} \to \mathfrak{X}(\mathscr{F})$, which associates to any $\tau \in R$ the fundamental vector field $\lambda(\tau)$. The variation of S induced by τ is then given by $\delta S(\tau) = \mathrm{d}S(\lambda(\tau))$. Notice that $\delta S(\ \ )$ can be seen as a map $\delta S(\ \ ) : \mathscr{K} \to \Omega_{\mathrm{loc}}^0$. The invariance of S

under $\mathscr{K}$ amounts to requiring that $\delta S(\tau) = 0$, for any $\tau \in k$.

We now consider the complex

$$\Omega^0_{\mathrm{loc}} \xrightarrow{\delta} \Omega^1_{\mathrm{loc}} \xrightarrow{\delta_1} \ldots \xrightarrow{\delta_{p-1}} \Omega^p_{\mathrm{loc}} \xrightarrow{\delta_p} \ldots$$

where Ω^p_{loc} is the space of p-linear skew-symmetric local maps $W : R\times \ldots \times R \to \Omega^0_{\mathrm{loc}}$ and $\delta_p W(\tau_1, \ldots, \tau_{p+1}) = \mathrm{d}\, W(\lambda(\tau_1), \ldots, \lambda(\tau_{p+1}))$. It is clear that $\delta_{p+1}\delta_p = 0$, so that we can construct the cohomology sets $H^p(R, \Omega^0_{\mathrm{loc}}) =: \ker \delta_p / \mathrm{im}\, \delta_{p-1}$. In the following we shall omit the subscripts and we will write simply δ for δ_p. As far as the relevance of this cohomology is concerned, note that, obviously enough, $\mathscr{K}$-invariant action functionals belong to $H^0(k, \Omega^0_{\mathrm{loc}})$. Indeed, the fact that $\delta S = 0$ amounts to state - via Noether's theorem - the existence of conserved currents. Conversely, conserved currents, once integrated over M, are cohomologous to 0 in $H^1(k, \Omega^0_{\mathrm{loc}})$.

Let us now consider a quantized version of the condition $\delta S(\ \ ) = 0$. We know that the existence of a quantum theory for a generic action S is not guaranteed; but, if the quantum action principle holds, then the quantum analogue Γ of the action should obey an equation of the form

$$\delta \Gamma = \hbar \Delta + 0(\hbar^2),$$

where $\Delta : k \to \Omega^0_{\mathrm{loc}}$. It is obvious that Δ should satisfy the consistency condition $\delta\Delta = 0$, so that it represents a cohomology class $[\Delta] \in H^1(k, \Omega^0_{\mathrm{loc}})$. If $[\Delta] \neq 0$, then it does not exist any function $\Gamma' \in \Omega^0_{\mathrm{loc}}$ such that $\Delta = \delta\Gamma'$. Accordingly, there is no redefinition $\widetilde{\Gamma} = \Gamma - \Gamma'$ of the action such that $\widetilde{\Gamma}$ is invariant under $\mathscr{K}$. We call any function Δ of this kind an anomaly.

As an example, consider the trace anomaly in the theory of a scalar field ϕ interacting with an external gravitational field $g_{\mu\nu}$ (see [25]). The classical action is

$$S(\phi, g) = \int_M \mathrm{d}^4 x \times \sqrt{g} \left(g^{\mu\nu}\ {}_\mu \phi \partial_\nu \phi - \frac{1}{6} R \phi^2 \right)$$

where R is the scalar curvature of $g_{\mu\nu}$. The action S is invariant under the conformal transformation $g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu}$, $\phi \to e^{-\sigma}\phi$, where σ is a real function on M. We can easily construct the coboundary operator relevant for this problem by setting $\sigma = \epsilon\xi$, where ϵ is a Grassmann (anticommuting) number and ζ is a Grassmann field. The coboundary operator δ is then represented by $\Xi = \Xi_g + \Xi_\phi$, where

$$\Xi_g = \int_M d^4x\, 2\xi\, g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}}$$

$$\Xi_\phi = -\int_M d^4x \xi\phi \frac{\delta}{\delta\phi}.$$

At the one-loop level, $\Xi\Gamma = \hbar\,\Delta + o(\hbar)$ and $\Xi\Delta = 0$. Since ϕ is not self-interacting, we can disregard ϕ-dependent terms and the most general function Δ with dimension 4 and «ghost» number 1 turns out to be

$$\Delta = \int_M d^4x\sqrt{g}\,\xi(a_1 R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\rho} + a_2 R_{\mu\nu}R^{\mu\nu} + a_3 R^2 + a_4 \Box R)$$

where a_i are real coefficients.

Now one has $\Xi\Delta = \int_M d^4 x \sqrt{g}\,(4a_1 + 4a_2 + 12a_3)\,R\,\xi\,\Box\,\xi$. Accordingly the condition $\Xi\Delta = 0$ implies that $a_1 + a_2 + 3a_3 = 0$, while a_4 is arbitrary. On the other hand, the first three terms of Δ cannot be obtained as Ξ-transform of anything. As far as the last term is concerned, one has $\Xi \int_M d^4 x\sqrt{g}\,\Box R = -2\int_M d^4 x \sqrt{g}\,\xi\,\Box R$ so the last term itself can be absorbed by a redefinition $\tilde{\Gamma}$ of Γ. The trace of the (anomalous) energy momentum tensor is thus

$$T^\mu_\mu = \frac{\delta}{\delta\xi}\,\Xi_g\tilde{\Gamma} = aF + bG$$

where

$$F = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\rho} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$$

$$G = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

and the constants a and b have to be determined by other methods.

This result extends to a self-interacting $\lambda\phi^4$ scalar field, as shown in detail by Bonora, Cotta-Ramusino and Reina ([25]), where we recover by means of cohomological techniques the general form of the anomalous terms considered by physicists. Other examples have been developed more recently, but we will not consider them here. For them, and further details we refer to our forthcoming paper with L. Bonora.

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